

A TALE OF TWO HECKE ALGEBRAS

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ABSTRACT. We use Bernstein's presentation of the Iwahori-Matsumoto Hecke algebra to obtain a simple proof of the Satake isomorphism and, in the same stroke, compute the center of the Iwahori-Matsumoto Hecke algebra.

1. INTRODUCTION

Let G be a connected, split, reductive group over a non-archimedean local field F . Fix a maximal split torus T in G . Then T determines a root system Φ . Let W be the corresponding Weyl group. Let K be a hyper-special maximal compact subgroup of G . More precisely, the torus T preserves a unique apartment in the Bruhat-Tits building of G , and we pick K to be the stabilizer of a hyper special vertex in the apartment. Then $T_K = T \cap K$ is a maximal compact subgroup of T , and the quotient $X = T/T_K$ is isomorphic to the co-character lattice of T . Let $H_K = C_c(K \backslash G / K)$ be the Hecke algebra of K -bi-invariant, compactly supported functions on G . Let $B = TN$ be a Borel subgroup containing T . Let $f \in C_c(G/K)$. Define $S(f)$, a function on T/T_K , by

$$S(f)(t) = \delta^{1/2}(t) \int_N f(tn) \, dn$$

where δ is the modular character. A famous theorem of Satake [Sa] states that the map S is an isomorphism of H_K and $\mathbb{C}[X]^W$.

Let $I \subset K$ be the Iwahori subgroup such that $I \cap B = K \cap B$. Let $H_I = C_c(I \backslash G / I)$ be the Hecke algebra of I -bi-invariant, compactly supported functions on G . Let Z_I be the center of H_I . The space $C_c(I \backslash G / K)$ is naturally a left H_I -module and a right H_K -module. Using Bernstein's description of H_I we show, in Theorem 1, that the map S gives an explicit isomorphism

$$S : C_c(I \backslash G / K) \rightarrow \mathbb{C}[X].$$

Then, as a simple consequence, we prove that the algebras Z_I , H_K and $\mathbb{C}[X]^W$ are isomorphic.

2. SOME PRELIMINARIES

The measure on G is normalized so that the volume of I is one. The space $C_c(G)$ of locally-constant, compactly supported functions is an algebra with respect to the convolution $*$ of functions. The unit of the algebra is H_K is denoted by 1_K . It is a function supported on K such that $1_K(k) = \frac{1}{[K:I]}$ for all $k \in K$.

For every root α we fix a homomorphism $\varphi_\alpha : \mathrm{SL}_2(F) \rightarrow G$. The co-root α^\vee is an element of X represented in T by

$$\varphi_\alpha \left(\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \right)$$

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where $v \in F$ has valuation 1. For every $u \in F$, let

$$x_\alpha(u) = \varphi_\alpha \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

We view the root α as a homomorphism $\alpha : X \rightarrow \mathbb{Z}$ such that, if $x \in X$ and $t_x \in T$ is a representative of x , then

$$t_x x_\alpha(u) t_x^{-1} = x_\alpha(vu)$$

where the valuation of v is $\alpha(x)$. We say that x is *dominant* if $\alpha(x) \geq 0$ for all positive roots α .

3. IWAHORI MATSUMOTO HECKE ALGEBRA

Let q be the order of the residue field of F . We summarize first some results of [IM].

The I -double co-sets in G are parameterized by $\tilde{W} = N_G(T_K)/T_K$. This group is a semi-direct product of the lattice X and the Weyl group W . The length function $\ell : \tilde{W} \rightarrow \mathbb{Z}$ is defined by

$$q^{\ell(w)} = [IwI : I].$$

Let T_w denote the characteristic function of the double coset IwI . Then $T_w T_v = T_{wv}$ if and only if $\ell(w) + \ell(v) = \ell(wv)$, and $\ell(w) + \ell(v) = \ell(wv)$ if and only if $IwIvI = IwvI$.

Let ρ be the sum of all positive roots. Then $\ell(x) = \rho(x)$ for a dominant $x \in X$. It follows that $T_x \cdot T_y = T_{x+y}$ for any two dominant x and y . Any $x \in X$ can be written as $x = y - z$ where y and z are two dominant elements in X . Following Bernstein, let

$$\theta_x = q^{(\ell(z) - \ell(y))/2} \cdot T_y T_z^{-1}.$$

Proposition 1. *Let $x \in X$, and $s \in W$ a reflection corresponding to a simple root α . Then*

$$T_s \theta_x - \theta_{s(x)} T_s = (1 - q) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha^\vee}}.$$

Lusztig [Lu] derives this proposition from [IM]. It can be also verified by a direct calculation in $\varphi_\alpha(\mathrm{SL}_2(F))$, see [S2].

Corollary 1. *Let $x \in X$, and $s \in W$ a simple reflection, as in Proposition 1. Then*

$$T_s(\theta_x + \theta_{s(x)}) = (\theta_x + \theta_{s(x)}) T_s.$$

Proposition 2. *(Bernstein's basis) Elements $\theta_x T_w$, where $x \in X$ and $w \in W$, form a basis of H_I .*

Proof. Since T_w , $w \in W$ and T_x , with x dominant generate H_I , Proposition 1 implies that $\theta_x T_w$ span H_I . Thus it remains to prove the linear independence. We follow an argument from [S1]. Assume that

$$\sum_{i,j} c_{i,j} \theta_{x_i} T_{w_j} = 0.$$

Let $x_0 \in X$ be dominant such that $x_0 + x_i$ is dominant for all x_i appearing in the sum. Then, after multiplying by θ_{x_0} from the left,

$$\sum_{i,j} c_{i,j} \theta_{x_0 + x_i} T_{w_j} = 0.$$

However, if x is dominant then $T_x \cdot T_w = T_{x \cdot w}$. In particular, $\theta_{x_0 + x_i} T_{w_j}$ are linearly independent. Thus $c_{i,j} = 0$. \square

Let A be the sub algebra of H_I generated by θ_x . Then $A \cong \mathbb{C}[X]$ via the isomorphism $\theta_x \mapsto [x]$. (We shall write an element in the group algebra $\mathbb{C}[X]$ as $\sum_{x \in X} c_x [x]$, where $c_x \in \mathbb{C}$, in order to distinguish $[x - y]$ from $[x] - [y]$.)

Proposition 3. *The centralizer of A in H_I is A .*

Proof. Let $z \in H_I$. Express z in the Bernstein's basis, and let $\theta_x T_w$ be a term in the expression such that $\ell(w)$ is maximal. If $w = 1$, then $z \in A$. Otherwise, there exists $y \in X$ such that $w(y) \neq y$. Now notice that $\theta_y \cdot \theta_x T_w = \theta_{y+x}$, while

$$\theta_x T_w \cdot \theta_y = \theta_{x+w(y)} T_w + \sum_{z,v} c_{z,v} \theta_z T_v$$

where $\ell(v) < \ell(w)$. As $y - w(y)$ can be made arbitrarily large, z does not commute with all elements in A . \square

4. SATAKE MAP

We fix the measure on N so that the volume of $(N \cap K)$ is $[K : I]$. We identify $C_c(T/T_K)$ with $\mathbb{C}[X]$ by $f \mapsto \sum_{x \in X} f(x)[x]$. The Satake map $S : C_c(G/K) \rightarrow C_c(T/T_K) = \mathbb{C}[X]$ is defined by

$$S(f)(t) = \delta(t)^{1/2} \int_N f(tn) \, dn.$$

It is a formal check (see [Ca]) that S , when restricted to $H_K = C_c(K \backslash G/K)$, is a homomorphism and the image of H_K is contained in $\mathbb{C}[X]^W$.

Proposition 4. *Let 1_K be the identity element of H_K . Then $\theta_x * 1_K$, $x \in X$, form a basis of $C_c(I \backslash G/K)$.*

Proof. Note that $C_c(I \backslash G/K) = C_c(I \backslash G/I) * 1_K$. Since $1_K = \frac{1}{[K:I]} \sum_{w \in W} T_w$, the proposition follows from Proposition 2. \square

Lemma 1. *Let (π, V) be a smooth G -module and (π', V') a smooth B -module with the trivial action of N . Let $S : V \rightarrow V'$ be a map such that $S(\pi(b)v) = \delta^{-1/2}(b)\pi'(b)S(v)$ for every $b \in B$. Then, for every $x \in X$ and $v \in V^I$,*

$$S(\pi(\theta_x)v) = \pi'(t_x)v.$$

This lemma appears in the literature in a special case when $V' = V_N$, the normalized Jacquet functor. The proof is the same and therefore omitted.

Theorem 1. *The map S induces an isomorphism of left $A \cong \mathbb{C}[X]$ -modules*

$$C_c(I \backslash G/K) \cong \mathbb{C}[X]$$

*which sends the basis elements $\theta_x * 1_K$ to the basis elements $[x]$.*

Proof. We apply Lemma 1 to $V = C_c(G/K)$, $V' = C_c(T/T_K)$ (considered as left G and T -modules) and S the Satake map. Then, for every $f \in C_c(I \backslash G/K)$, $S(\theta_x * f)(t) = S(f)(t_x^{-1}t)$. Thus $S(\theta_x * f) = [x] \cdot S(f)$. In particular, $S(\theta_x * 1_K) = [x] \cdot S(1_K) = [x]$, and the theorem follows. \square

Let Z_I be the center of H_I . Let A^W be the span of $\sum_{w \in W} \theta_{w(x)}$ for $x \in X$. Corollary 1 implies that $A^W \subseteq Z_I$. Let $Z : Z_I \rightarrow H_K$ be a homomorphism defined by $Z(z) = z * 1_K$.

Theorem 2. *The maps Z and S induce isomorphisms of algebras*

$$A^W \cong Z_I \cong H_K \cong \mathbb{C}[X]^W.$$

Proof. Theorem 1 implies that S , restricted to H_K , is injective. Proposition 3 implies that $Z_I \subseteq A$. This and Theorem 1 imply that the map $S \circ Z$ is injective. Thus, we have the injections

$$A^W \subseteq Z_I \subseteq H_K \subseteq \mathbb{C}[X]^W.$$

Since $(S \circ Z)(\sum_{w \in W} \theta_{w(x)}) = S(\sum_{w \in W} \theta_{w(x)} * 1_K) = \sum_{w \in W} [w(x)]$, the above injections are isomorphisms. □

Final Remarks. A proof of the isomorphism $Z_I \cong \mathbb{C}[X]^W$ can be found in [Da] and [HKP]. Both approaches are based on the explicit description of the Bernstein component of the category of smooth G -modules containing the trivial representation. Dat also shows that the map Z gives an isomorphism of Z_I and H_K . On the other hand, Lusztig [Lu] considers a version of the algebra H_I over the ring $\mathbb{Z}[q^{\pm 1/2}]$ where q is considered a formal variable. He shows that the center is isomorphic to $\mathbb{Z}[q^{\pm 1/2}][X]^W$ by specializing $q^{1/2} = 1$. No claim is made as to what the center is when q is specialized to a power of a prime number.

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